Mathematical Methods in Physics HW7

1. Consider polynomials on the interval $x \in [0, \infty)$ with inner product $(f, g) = \int_0^\infty f^*(x) g(x) e^{-x}\, dx$.

   a) Gram-Schmidt the linearly independent set $\{x^0, x^1, x^2, \ldots\}$ to find the first three orthonormal polynomials $\hat{L}_n(x)$.

   $$\hat{L}_0(x) = \frac{x^0}{\|x^0\|} = \frac{1}{\|1\|} = \frac{1}{\sqrt{(1,1)}} = \frac{1}{\sqrt{\int_0^\infty 1 e^{-x} dx}} = 1$$

   $$\hat{L}_1(x) = \frac{x - \hat{L}_0(\hat{L}_0, x)}{\|x - \hat{L}_0(\hat{L}_0, x)\|} = \frac{x-1}{\sqrt{\int_0^\infty (x-1)^2 e^{-x} dx}} = x - 1$$

   $$\hat{L}_2(x) = \frac{x^2 - \hat{L}_1(\hat{L}_1, x^2) - \hat{L}_0(\hat{L}_0, x^2)}{\|x^2 - \hat{L}_1(\hat{L}_1, x^2) - \hat{L}_0(\hat{L}_0, x^2)\|} = \frac{x^2 - (x-1)(4) - 1(2)}{\|x^2 - (x-1)(4) - 1(2)\|} = \frac{x^2 - 4x + 2}{\sqrt{\int_0^\infty (x^2 - 4x + 2)^2 e^{-x} dx}} = \frac{1}{2}(x^2 - 4x + 2)$$

   b) Make the first three orthogonal polynomials with the Rodrigues formula $L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(e^{-x} x^n)$. What are the normalization factors?

   $$L_0(x) = \frac{e^x}{0!}(e^{-x} x^0) = 1$$

   $$L_1(x) = \frac{e^x}{1!} \frac{d}{dx}(e^{-x} x) = e^x(-e^{-x} x + e^{-x}) = 1 - x$$

   $$L_2(x) = \frac{e^x}{2!} \frac{d^2}{dx^2}(e^{-x} x^2) = \frac{e^x}{2} \frac{d}{dx}(-e^{-x} x^2 + 2e^{-x} x)$$

   $$= \frac{e^x}{2}(e^{-x} x^2 - 2e^{-x} x - 2e^{-x} x + 2e^{-x}) = \frac{1}{2}(x^2 - 4x + 2)$$

   The Rodrigues formula is already giving normalized functions.

   c) Now consider the generating function $\varphi(x, t) = \frac{1}{1-t} e^{\frac{-xt}{1-t}} = \sum_{n=0}^\infty \frac{t^n}{n!} L_n(x)$. Following some of the steps that we did in class (differentiating this w.r.t. to x and t then peeling off expressions that result from grabbing everything in front of t^n), you should generate two equations that can then be manipulated to derive the differential equation for which the Laguerre polynomials are solutions, i.e. $x L_n''(x) + (1 - x) L_n'(x) + n L_n(x) = 0$. Try this. Definitely get the two equations mentioned, and you can try to combine them to get the differential equation, but it can be quite tricky.

   $$\frac{d\varphi}{dx} = -\frac{t}{1-t}\varphi \;\Rightarrow\; \sum \frac{t^n}{n!} L_n' = -\frac{t}{1-t} \sum \frac{t^n}{n!} L_n \;\Rightarrow\; \sum \frac{t^n}{n!} L_n' - \sum \frac{t^{n+1}}{n!} L_n' = -\sum \frac{t^{n+1}}{n!} L_n$$

   $$t^n: \; \frac{1}{n!} L_n' - \frac{1}{(n-1)!} L_{n-1}' = -\frac{1}{(n-1)!} L_{n-1} \quad \Rightarrow \quad L_n' - n L_{n-1}' = -n L_{n-1} \quad = \text{Eqn\#1}$$

   $$\frac{d\varphi}{dt} = \frac{1 - x - t}{1 - 2t + t^2}\varphi \;\Rightarrow\; \sum \frac{n t^{n-1}}{n!} L_n = \frac{1 - x - t}{1 - 2t + t^2} \sum \frac{t^n}{n!} L_n$$

   $$\Rightarrow \sum \frac{n t^{n-1}}{n!} L_n - 2 \sum \frac{t^n}{n!} L_n + \sum \frac{n t^{n+1}}{n!} L_n = \sum \frac{t^n}{n!} L_n - x \sum \frac{t^n}{n!} L_n - \sum \frac{t^{n+1}}{n!} L_n$$

   $$t^n: \; \frac{n+1}{(n+1)!} L_{n+1} - 2\frac{n}{n!} L_n + \frac{n-1}{(n-1)!} L_{n-1} = \frac{1}{n!} L_n - \frac{x}{n!} L_n - \frac{1}{(n-1)!} L_{n-1}$$

   $$\Rightarrow \quad L_{n+1} - 2n L_n + n(n-1) L_{n-1} = L_n - x L_n - n L_{n-1}$$

   $$\Rightarrow \quad L_{n+1} + (x - 2n - 1) L_n + n^2 L_{n-1} = 0 \quad = \text{Eqn\#2}$$

   Start by taking Eqn#1 and shifting it from $n \to n + 1$

   $$L_{n+1}' - (n+1) L_n' = -(n+1) L_n \qquad = \text{Eqn\#3}$$

   and then taking the derivative of it:

   $$L_{n+1}'' - (n+1) L_n'' = -(n+1) L_n' \qquad = \text{Eqn\#4}$$

Now shift the previous result from $n+1 \to n+2$

$L''_{n+2} - (n+2)L''_{n+1} = -(n+2)L'_{n+1}$

and now use Eqn#3 to replace the L'_{n+1} term leading to:

$L''_{n+2} - (n+2)L''_{n+1} + (n^2+3n+2)L'_n - (n^2+3n+2)L_n = 0$ = Eqn#5

Now use Eqn#2 and take the second derivative and bump it up to $n+1 \to n+2$

$L''_{n+2} + (x-2n-3)L''_{n+1} + 2L'_{n+1} + (n+1)^2 L''_n = 0$

Now looking at it, realize that we can take Eqn#5 to replace L''_{n+2} in the preceding with L''_{n+1}, L'_n, L_n. Then we can use Eqn#4 to replace all of theL''_{n+1} with L''_n, L'_n. And lastly we can use Eqn#3 to replace the L'_{n+1} with L'_n, L_n. This gives everything in terms of L''_n, L'_n, L_n.

Doing so gives:

$xL''_n + (1-x)L'_n + nL_n = 0$

2. Consider starting with an operator $L = (\alpha_o x^2 + \alpha_1 x + \alpha_2)\frac{d^2}{dx^2} + (\beta_0 x + \beta_1)\frac{d}{dx}$. Use the conditions 1-3 from class to reduce this to a form such that $\alpha_0 \neq 0$ and the interval over which this is Hermitian is $[a,b]$. Determine the weight that would be used in the inner product in this case.

We want $\alpha(x) = (x-a)(x-b) = x^2 - (a+b)x + ab$ so that the roots are at $x = a$ and $x = b$.

Starting with $\alpha_o x^2 + \alpha_1 x + \alpha_2$ we can rescale the overall polynomial with k, then shift the variable x by j, and finally rescale the newly shifted variable $x + j$ by m. The result is:

$k\alpha_0(mx+mj)^2 + k\alpha_1(mx+mj) + k\alpha_2$

$= (m^2 k\alpha_0)x^2 + (2m^2 k\alpha_0 j + mk\alpha_1)x + (m^2 k\alpha_0 j^2 + mk\alpha_1 j + k\alpha_2)$

We want this to equal $x^2 - (a+b)x + ab$, so pairing up coefficients of the same order in x we find:

Eqn#1: $m^2 k\alpha_0 = 1$

Eqn#2: $2m^2 k\alpha_0 j + mk\alpha_1 = -(a+b)$

Eqn#3: $m^2 k\alpha_0 j^2 + mk\alpha_1 j + k\alpha_2 = ab$

Eqn#1 $\Rightarrow$ $k = \frac{1}{\alpha_0 m^2}$, inserted into Eqn#21 $\Rightarrow$ $2j + \frac{\alpha_1}{m\alpha_0} = -(a+b)$ $\Rightarrow$ $j = -\frac{(a+b)}{2} - \frac{\alpha_1}{2m\alpha_0}$

Plugging the expressions for k, j into Eqn#3 gives:

$m^2 = \frac{\alpha_1^2 - 4\alpha_2\alpha_0}{\alpha_0^2(a-b)^2}$ $\Rightarrow$ $m = \frac{\sqrt{\alpha_1^2-4\alpha_2\alpha_0}}{\alpha_0(a-b)}$ $\Rightarrow$ $k = \frac{\alpha_0(a-b)^2}{\alpha_1^2-4\alpha_2\alpha_0}$ and $j = -\frac{(a+b)}{2} - \frac{\alpha_1}{2\alpha_0}\frac{\alpha_0(a-b)}{\sqrt{\alpha_1^2-4\alpha_2\alpha_0}}$

Plugging these solutions in we find that $\alpha(x) = (x-a)(x-b)$ as needed.

To find $w(x)$ we first redefine $\beta_0 = q + p + 2$ and $\beta_1 = -qb - pa - b - a$ and then use:

$w(x)\alpha(x) = Ce^{\int \frac{\beta}{\alpha}dx} = Ce^{\int\left(\frac{q+1}{x-a}+\frac{p+1}{x-b}\right)dx} = Ce^{(q+1)\ln(x-a)+(p+1)\ln(x-b)} = C(x-a)^{q+1}(x-b)^{p+1}$

$w(x) = C(x-a)^q(x-b)^p$

3. Find an expression for the Fourier transform of a product of three transformable functions $f_1(y), f_2(y), f_3(y)$ in terms of their transforms $g_1(k), g_2(k), g_3(k)$.

$$G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_1(y) f_2(y) f_3(y) e^{-iky} dy$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_1(k') e^{ik'y} dk'\right) f_2(y) f_3(y) e^{-iky} dy$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_1(k') \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(y) f_3(y) e^{ik'y-iky} dy\right) dk'$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_1(k') \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_2(y) f_3(y) e^{-ik''y} dy\right) dk' \quad \text{where } k'' = k - k'$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_1(k') \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g_2(k''') g_3(k'' - k''') dk'''\right) dk'$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(k') g_2(k''') g_3(k - k' - k''') dk''' dk'$$

4. Using the result of problem 1, evaluate the Fourier transform of the product of the three functions $f_1(y) = e^{ik_1 y}$, $f_2(y) = e^{ik_2 y}$ and $f_3(y) = e^{ik_3 y}$. Verify that your answer makes sense by considering the product as a single function.

First of all $g_i(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_i(y) e^{-iky}\, dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(k_i - k)y}\, dy = \sqrt{2\pi}\,\delta(k_i - k)$

$$G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{k_1 y} e^{k_2 y} e^{k_3 y} e^{-iky} dy$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{2\pi}\,\delta(k_1 - k')\,\sqrt{2\pi}\,\delta(k_2 - k''')\sqrt{2\pi}\,\delta(k_3 - k + k' + k''') dk''' dk'$$

$$= 2\pi\delta(k_3 - k + k_1 + k_2) = 2\pi\delta(k_1 + k_2 + k_3 - k)$$

Alternatively,

$g(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(k_1+k_2+k_3)} e^{-iky} dy = 2\pi\delta(k_1 + k_2 + k_3 - k)$ as expected.

5. Find Y_{20} by Gram-Schmidting.

For $l = 2$ and $m = 0$ we have $f_{20}(\cos\theta) = \bar{Y}_{20} = A + B\cos\theta + C\cos^2\theta$.

Then $Y_{20} = \frac{\bar{Y}_{20} - Y_{00}(Y_{00},\bar{Y}_{20}) - Y_{10}(Y_{10},\bar{Y}_{20}) - Y_{1-1}(Y_{1-1},\bar{Y}_{20}) - Y_{11}(Y_{11},\bar{Y}_{20})}{\|\bar{Y}_{20} - Y_{00}(Y_{00},\bar{Y}_{20}) - Y_{10}(Y_{10},\bar{Y}_{20}) - Y_{1-1}(Y_{1-1},\bar{Y}_{20}) - Y_{11}(Y_{11},\bar{Y}_{20})\|}$

Doing these separately and using that:

$$\int_0^{\pi} \sin\theta \cos^m\theta\, d\theta = -\frac{1}{m+1}(\cos^{m+1}\pi - 1)$$

$\int_0^{\pi} \cos^m\theta\, d\theta = \frac{m-1}{m}\int_0^{\pi} \cos^{m-2}\theta\, d\theta$ for $n > 1$

$\int_0^{2\pi} e^{\pm i\varphi} d\varphi = 0$

$Y_{00}(Y_{00}, \bar{Y}_{20}) = \frac{1}{2\sqrt{\pi}} \int_0^{2\pi} \int_0^{\pi} \frac{1}{2\sqrt{\pi}} \sin\theta\,(A + B\cos\theta + C\cos^2\theta) d\theta d\varphi$

$= \frac{2\pi}{4\pi}\left(\int_0^{\pi} [A\sin\theta + B\sin\theta\cos\theta + C\sin\theta\cos^2\theta] d\theta\right)$

$= \frac{1}{2}\left(2A + 0 + \frac{2}{3}C\right) = A + \frac{1}{3}C$

$Y_{10}(Y_{10}, \bar{Y}_{20}) = \frac{\sqrt{3}}{2\sqrt{\pi}} \cos\theta \int_0^{2\pi} \int_0^{\pi} \frac{\sqrt{3}}{2\sqrt{\pi}} \sin\theta\cos\theta\,(A + B\cos\theta + C\cos^2\theta) d\theta d\varphi$

$= \frac{6\pi}{4\pi} \cos\theta \left(\int_0^{\pi} [A\sin\theta\cos\theta + B\sin\theta\cos^2\theta + C\sin\theta\cos^3\theta]\, d\theta\right)$

$= \frac{3}{2}\cos\theta \left(0 + \frac{2}{3}B + 0\right) = B\cos\theta$

$Y_{1-1}(Y_{1-1}, \bar{Y}_{20}) = \frac{\sqrt{3}}{2\sqrt{2\pi}} e^{-i\varphi} \sin\theta \int_0^{2\pi} \int_0^{\pi} \frac{\sqrt{3}}{2\sqrt{2\pi}} e^{i\varphi} \sin^2\theta\,(A + B\cos\theta + C\cos^2\theta) d\theta d\varphi$

$= \frac{3}{8\pi} e^{-i\varphi} \sin\theta \left(\int_0^{2\pi} e^{i\varphi}\, d\varphi \int_0^{\pi} [A\sin^2\theta + B\sin^2\theta\cos\theta + C\sin^2\theta\cos^2\theta]\, d\theta\right)$

$= 0$

$Y_{11}(Y_{11}, \bar{Y}_{20}) = -\frac{\sqrt{3}}{2\sqrt{2\pi}} e^{i\varphi} \sin\theta \int_0^{2\pi} \int_0^{\pi} -\frac{\sqrt{3}}{2\sqrt{2\pi}} e^{-i\varphi} \sin^2\theta\,(A + B\cos\theta + C\cos^2\theta) d\theta d\varphi$

$= \frac{3}{8\pi} e^{i\varphi} \sin\theta \left(\int_0^{2\pi} e^{-i\varphi}\, d\varphi \int_0^{\pi} [A\sin^2\theta + B\sin^2\theta\cos\theta + C\sin^2\theta\cos^2\theta]\, d\theta\right)$

$= 0$

Finally:

$$Y_{20} = \frac{A + B\cos\theta + C\cos^2\theta - A - \frac{1}{3}C - B\cos\theta}{\left\| A + B\cos\theta + C\cos^2\theta - A - \frac{1}{3}C - B\cos\theta \right\|} = \frac{\cos^2\theta - \frac{1}{3}}{\left\|\cos^2\theta - \frac{1}{3}\right\|} = \frac{\cos^2\theta - \frac{1}{3}}{\sqrt{\int_0^{2\pi}\int_0^{\pi} \sin\theta\left(\cos^2\theta - \frac{1}{3}\right)^2 d\theta d\varphi}} = \sqrt{\frac{45}{16\pi}}\left(\cos^2\theta - \frac{1}{3}\right)$$

6. Find Y_{32} using whatever method you want.

We can Rodrigues it to find:

$Y_{32}(\theta, \varphi) = (-1)^2 \left[\frac{7}{4\pi}\frac{1!}{5!}\right]^{\frac{1}{2}} P_3^2(\cos\theta) e^{i2\varphi}$

Where $P_3^2(x) = (1 - x^2)\frac{d^2}{dx^2} P_3(x) = (1 - x^2)\frac{d^2}{dx^2}\left[\frac{1}{2}(5x^3 - 3x)\right] = 15(x - x^3)$

Then $Y_{32}(\theta, \varphi) = 15\left[\frac{7}{480\pi}\right]^{\frac{1}{2}} (\cos\theta - \cos^3\theta) e^{i2\varphi} = \frac{1}{4}\sqrt{\frac{105}{2\pi}} \sin^2\theta\cos\theta\, e^{i2\varphi}$

7. Evaluate the Stieltjes integral of $f(x) = e^{-x}$ with the measure inducing function given by $g(x) = n$ for $n \leq x < n+1$ where $n \in \mathbb{Z}$ (integers that is) over $x \in [0, \infty]$.

$\int_0^\infty f(x)dg(x) = \lim_{x_{i+1}-x_i \to 0} \sum_{i=0}^{\infty} f(\bar{x}_i)[g(x_{i+1}) - g(x_i)] = \sum_{i=0}^{\infty} e^{-i} = 1 + \frac{1}{e-1} = \frac{e}{e-1}$

8. Verify that the resolution of the identity for the position operator X, i.e. $Xf(x) = xf(x)$, is given by $E(\mu)f(x) = \begin{cases} f(x) & x \leq \mu \\ 0 & x > \mu \end{cases}$.

First we show that $E(\mu)$ satisfies the usual requirements:

Obviously $E(-\infty) = 0$ since any $x > -\infty$, and $E(\infty) = I$ since any $x \leq \infty$.

Now we need to show that $E(\mu_1)E(\mu_2) = E(\mu_1)$ is $\mu_1 < \mu_2$:

$E(\mu_1)E(\mu_2)f(x) = 0 = E(\mu_1)f(x)$ for $\mu_1, \mu_2 < x$

$E(\mu_1)E(\mu_2)f(x) = 0 = E(\mu_1)f(x)$ for $\mu_1 < x < \mu_2$

$E(\mu_1)E(\mu_2)f(x) = f(x) = E(\mu_1)f(x)$ for $\mu_1, \mu_2 > x$

Finally, $\int_{-\infty}^{\infty} dE(\mu)\, f(x) = \lim_{\Delta \to 0} \sum_{i=0}^{\infty} [E(\mu_i + \Delta) - E(\mu_i)]\, f(x)$. Now for each x, only one value of i will give $E(\mu_i + \Delta) = I$ while $E(\mu_i) = 0$, and it is when $\mu_i = x$. But this means that for each value of x, we get that this is acting as the identity at that value, but it works for any value of x, i.e. $\int_{-\infty}^{\infty} dE(\mu)\, f(x) = If(x)$.

Now we should show that its action is $\left[\int_{-\infty}^{\infty} \mu dE(\mu)\right] f(x) = xf(x)$, but using the interpretation above, it should be obvious that:

$\left[\int_{-\infty}^{\infty} \mu dE(\mu)\right] f(x) = xf(x)$, i.e. $\left[\int_{-\infty}^{\infty} \mu dE(\mu)\right] = X$.

9. Given that $E_A(\lambda)$ is the resolution of the identity for the operator A. Show that $E_A(\lambda_2) - E_A(\lambda_1) = [E_A(\lambda_2) - E_A(\lambda_1)]^2$ for $\lambda_2 > \lambda_1$ but not for $\lambda_1 > \lambda_2$.

$$[E_A(\lambda_2) - E_A(\lambda_1)]^2 = E_A(\lambda_2)^2 - E_A(\lambda_2)E_A(\lambda_1) - E_A(\lambda_1)E_A(\lambda_2) + E_A(\lambda_1)^2$$

$$= E_A(\lambda_2) - E_A(\lambda_1) - E_A(\lambda_1) + E_A(\lambda_1) = E_A(\lambda_2) - E_A(\lambda_1)$$

Note that this doesn't work when $\lambda_1 > \lambda_2, i.e.$

$$[E_A(\lambda_2) - E_A(\lambda_1)]^2 = E_A(\lambda_2)^2 - E_A(\lambda_2)E_A(\lambda_1) - E_A(\lambda_1)E_A(\lambda_2) + E_A(\lambda_1)^2$$

$$= E_A(\lambda_2) - E_A(\lambda_2) - E_A(\lambda_2) + E_A(\lambda_1) = -E_A(\lambda_2) + E_A(\lambda_1) \neq E_A(\lambda_2) - E_A(\lambda_1)$$

10. Consider the quantum-mechanical observables L_x, L_y and L_z, which are represented as matrices as follows:

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The Hamiltonian for this system has the form $H = H_0 + L_z$ where H_0 is time-independent and commutes with L_x, L_y and L_z.

a) Supposer that at $t = 0$ we have prepared the system in an eigenstate of L_z, namely, the state belonging to the eigenvalues $m = +1$ of L_z. If we measure L_z at a later time $t = T$, what are the probabilities that we will find the values $+1$,0 or -1.

Let's start with the eigens of L_z:

$$det(L_z - \lambda I) = det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -1-\lambda \end{pmatrix} = 0 \quad \Rightarrow \quad \lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -1$$

$$L_z\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ -c \end{pmatrix} = \lambda_1\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$L_z\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ -c \end{pmatrix} = \lambda_2\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$L_z\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ -c \end{pmatrix} = \lambda_3\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \end{pmatrix} \quad \Rightarrow \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Since H_0 is time-independent, we know that since $[H_0 + L_z, L_z] = 0$, then the probabilities of the results of measurement of L_z do not change with time. But since we are starting at $t = 0$ in the state v_1 with $\lambda = \lambda_1 = +1$, then this will have 100% probability at time $t = T$, and the probability of getting 0 or -1 as the results is zero.

b) Suppose instead that we measure L_x at time $t = T$. What are the possible values of L_x that can be obtained and what are their probabilities?

First of all, note that $[H_0 + L_z, L_z] = 0$ then measuring L_z at $t = 0$ and getting v_1 means it will remain in this state unless and until we make another measurement. If we make a measurement of L_x at time $t = T$, it is just like measuring L_x for an eigenstate of L_z.

First the L_x eigen story is:

$$det(L_x - \lambda' I) = det\begin{pmatrix} -\lambda' & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda' & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda' \end{pmatrix} = -\lambda'\left(\lambda'^2 - \frac{1}{2}\right) + \frac{1}{2}\lambda' = 0 \quad \Rightarrow \quad \lambda'_1 = 1, \lambda'_2 = 0, \lambda'_3 = -1$$

$$L_x\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}b \\ \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}c \\ \frac{1}{\sqrt{2}}b \end{pmatrix} = \lambda'_1\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \Rightarrow \quad v'_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

$$L_x\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}b \\ \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}c \\ \frac{1}{\sqrt{2}}b \end{pmatrix} = \lambda'_2\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad v'_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$L_x\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}b \\ \frac{1}{\sqrt{2}}a + \frac{1}{\sqrt{2}}c \\ \frac{1}{\sqrt{2}}b \end{pmatrix} = \lambda'_3\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -a \\ -b \\ -c \end{pmatrix} \quad \Rightarrow \quad v'_3 = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

To determine the probabilities of getting each of these L_x eigenvalues as a result of a measurement done on a system in the state v_1, we can simply use:

$$P(\lambda'_i) = |(v'_i, v_1)|^2 \quad \Rightarrow \quad P(\lambda'_1) = |(v'_1, v_1)|^2 = \frac{1}{4}, P(\lambda'_2) = |(v'_2, v_1)|^2 = \frac{1}{2}, P(\lambda'_3) = |(v'_3, v_1)|^2 = \frac{1}{4}$$

Which of course satisfy $\sum_i P_i = 1$.